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Fault Identification: An Approach Based on Optimal Control Methods

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Abstract—This paper considers the problem of identifying (estimating) faults in systems described by linear models under exogenous disturbances. It is solved using optimal control methods; in comparison with sliding mode observers, they avoid high-frequency switching. The solution method proposed below involves a reduced model of the original system that is sensitive to faults and insensitive to disturbances. The corresponding theory is illustrated by an example.

Keywords: linear systems, faults, identification, observers, optimal systems

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1. INTRODUCTION

For the last two decades, the problem of fault identification has been solved based on sliding mode observers [1–7]. In the works cited, certain constraints were imposed on the system under consideration. The most typical ones include the matching condition and the minimum phase property of the system. This restricts the class of systems for which such observers can be constructed. In addition, the implementation of such observers implies high-frequency switching and, consequently, high-frequency data exchange in the control system, which is not always practicable. The method based on the optimal control theory proposed below is free from this disadvantage.

Consider control systems described by the linear model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dd(t) + L\rho(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t) \end{aligned} \tag{1.1}$$

with the following notations: $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, and $y \in \mathbb{R}^l$ is the output; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $D \in \mathbb{R}^{n\times q}$, and $L \in \mathbb{R}^{n\times s}$ are known constant matrices; the vector function $d(t) \in \mathbb{R}^q$ describes faults, i.e., $d(t) = 0$ if there are no faults, and $d(t)$ becomes an unknown time-varying function otherwise; finally, $\rho(t) \in \mathbb{R}^s$ is an unknown time-varying function of exogenous disturbances affecting the system.

In this paper, the problem is to design an observer for estimating the function $d(t)$. In contrast to the conventional approach, the solution proposed below is based on optimal control methods. By analogy with [5–7], the problem is solved not for the original system but for its reduced model insensitive to disturbances. Such a model has a smaller dimension than the original system.

2. BUILDING THE REDUCED MODEL

The reduced model has the form

$$\begin{aligned} \dot{x}_*(t) &= A_* x_*(t) + B_* u(t) + J_* y(t) + D_* d(t), \\ y_*(t) &= C_* x_*(t), \end{aligned} \tag{2.1}$$

where $x_*(t) \in \mathbb{R}^k$ is the state vector; A_*, B_*, J_*, C_*, and D_* are matrices of compatible dimensions to be determined. By analogy with [5–7], the matrices A_* and C_* are found in the canonical form

$$A_* = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad C_* = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{2.2}$$

Also following [5–7], we suppose the existence of matrices Φ and R_* such that $x_*(t) = \Phi x(t)$, $y_*(t) = R_* y(t)$, and

$$\Phi A = A_* \Phi + J_* C, \quad R_* C = C_* \Phi, \quad \Phi B = B_*, \quad \Phi D = D_*. \tag{2.3}$$

In view of the canonical form (2.2), equations (2.3) imply [5–7] the equations

$$\begin{aligned} &\Phi_1 = R_* C, \quad \Phi_i A = \Phi_{i+1} + J_{*i} C, \quad i = 2, \ldots, k-1, \\ &\Phi_k A = J_{*k} C, \end{aligned} \tag{2.4}$$

where Φ_i and J_{*i} are the ith rows of the matrices Φ and J_*, respectively, $i = 1, \ldots, k$. The matrix R_* must be chosen so that $D_* \neq 0$. The corresponding procedure will be given below.

Assumption 1. $\mathrm{Im}(D) \not\subset \mathrm{Ker}(V^{(n)})$, where

$$V^{(n)} = \begin{pmatrix} C \\ CA \\ \ldots \\ CA^{n-l} \end{pmatrix}$$

is the observability matrix.

Assumption 1 holds if system (1.1) is observable: in this case, $\mathrm{Ker}(V^{(n)}) = 0$ and, consequently, $V^{(n)} D \neq 0$. Let p be the smallest integer satisfying $CA^p D \neq 0$ and j be an integer for which $C_j A^p D \neq 0$. It can be demonstrated that (2.4) implies $\Phi = Q V^{(n)}$ for some matrix Q, and then we obtain $D_* = \Phi D \neq 0$ from $C_j A^p D \neq 0$. According to the aforesaid, the pth derivative of the variable y_j is sensitive to faults: this derivative will change value if a fault occurs. Also, obviously, if the jth position of the matrix R_* is nonzero, model (2.1) with this matrix will be sensitive to faults.

As was shown in [5–7], insensitivity to exogenous disturbances holds if $\Phi L = 0$. Together with (2.4), this condition can be reduced to the equation

$$\begin{pmatrix} R_* & -J_{*1} & \ldots & -J_{*k} \end{pmatrix} (W^{(k)} \; L^{(k)}) = 0, \tag{2.5}$$

where

$$W^{(k)} = \begin{pmatrix} CA^k \\ CA^{k-1} \\ \ldots \\ C \end{pmatrix}, \quad L^{(k)} = \begin{pmatrix} CL & CAL & \ldots & CA^{k-1}L \\ 0 & CL & \ldots & CA^{k-2}L \\ & & \ldots & \\ 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Equation (2.5) has a solution under

$$\mathrm{rank}\,(W^{(k)} \quad L^{(k)}) < l(k+1).$$

This inequality serves to determine the minimal dimension $k > p$; equation (2.5), to determine the row $(R_* \ \ -J_{*1} \ \ \ldots \ \ -J_{*k})$. If the jth position of the matrix R_* is nonzero, then the matrices Φ, B_*, and D_* are obtained using (2.3) and (2.4). Otherwise, it is necessary to find another solution of equation (2.5).

The stability of the model is ensured by feedback on the residual signal $r_*(t) = R_* y(t) - y_*(t)$:

$$\dot{x}_*(t) = A_* x_*(t) + B_* u(t) + J_* y(t) + D_* d(t) + K r_*(t), \tag{2.6}$$

where the matrix K has the form $K = (k_1 \ \ k_2 \ \ \ldots \ \ k_k)^{\mathrm{T}}$. The coefficients $k_1, k_2, \ldots, k_k$ are determined from given eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$:

$$\begin{aligned} k_1 &= -(\lambda_1 + \lambda_2 + \ldots + \lambda_k), \\ k_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{k-1}\lambda_k, \\ &\ldots, \\ k_k &= (-1)^k \lambda_1 \lambda_2 \ldots \lambda_k. \end{aligned}$$

Consider the expression for the residual $r_*(t)$, equation (2.6) can be transformed into

$$\dot{x}_*(t) = (A_* - KC_*) x_*(t) + B_* u(t) + (J_* + KR_*) y(t) + D_* d(t).$$

3. AN AUXILIARY OPTIMAL CONTROL PROBLEM

As has been emphasized, the problem of fault identification is solved using optimal control methods. Consider the corresponding problem for the system

$$\begin{aligned} &\dot{z}(t) = (A_* - KC_*) z(t) + B_* u(t) + (J_* + KR_*) y(t) + D_* w(t), \ \ z(t_0) = \Phi x_0, \\ &y_z(t) = C_* z(t), \end{aligned} \tag{3.1}$$

where the auxiliary control variable $w(t)$ plays the role of the unknown function $d(t)$. It is chosen to transfer system (3.1) from the state $z(t_0)$ to a target state with the output $y_z(t_f)$ such that $y_z(t_f) \to y_*(t_f)$ as $t_f \to \infty$ and

$$J = \frac{1}{2} \int\limits_{t_0}^{\infty} \left(e_y^{\mathrm{T}} Q e_y + w^{\mathrm{T}} R w \right) dt \to \min_{v}. \tag{3.2}$$

Here, $e_y(t) = y_z(t) - y_*(t)$ denotes the residual, and Q and $R \in \mathbb{R}^{q \times q}$ are a positive number and a positive definite matrix, respectively. The relation $y_z(t_f) \to y_*(t_f)$, $t_f \to \infty$, is understood as convergence in the Euclidean norm: $\| y_z(t_f) - y_*(t_f) \| \to 0$ as $t_f \to \infty$. The convergence of other time-varying functions in this paper is interpreted by analogy.

The identification problem is to construct the optimal control $w(t)$ in the sense of the performance criterion (3.2) such that $y_z(t) \to y_*(t)$ and $w(t) \to d(t)$ as $t \to \infty$. The criterion (3.2) must be minimized for a sufficiently large value of the constant Q, in particular, $Q = 10^{20}$ in the example below. This practically ensures the property $e_y(t) \to 0$ as $t \to \infty$. With this in mind, we denote by e_{y*} the asymptote for $e_y(t)$. According to the previous considerations, $e_{y*} = 0$ can be taken with a sufficient degree of accuracy.

Introducing the error vector $e(t) = z(t) - x_*(t) \in R^k$, we write the corresponding equation

$$\begin{aligned} \dot{e}(t) &= A_* e(t) + D_*(w(t) - d(t)) - K e_y(t) \\ &= (A_* - KC_*) e(t) + D_*(w(t) - d(t)), \quad e(t_0) = 0, \\ e_y(t) &= C_* e(t). \end{aligned} \tag{3.3}$$

Assumption 2. System (3.3) is strongly observable.

Strong observability means the absence of invariant zeros. In other words, there exist no s for which

$$\operatorname{rank}(R(s)) < k + \operatorname{rank}\begin{pmatrix} -D_* \\ 0 \end{pmatrix},$$

where $R(s)$ is the Rosenbrock matrix [8, 9]:

$$R(s) = \begin{pmatrix} sI - (A_* - KC_*) & -D_* \\ C_* & 0 \end{pmatrix}.$$

Theorem 1. *If system* (3.3) *is strongly observable, then* $e_y(t) \to 0$ *implies* $w(t) \to d(t)$ *as* $t \to \infty$.

Proof. Let $H(s)$ be the transfer function of system (3.3):

$$E_y(s) = H(s)(W(s) - D(s)), \tag{3.4}$$

where $E_y(s)$, $W(s)$, and $D(s)$ are the Laplace images of the functions e_y, $w(t)$, and $d(t)$, and s denotes the complex variable. Since $e_{y*} = 0$, it follows that $E_y(s) = 0$. System (3.3) has no invariant zeroes; hence, for all s, the function $H(s)$ is nonzero and, consequently, $W(s) = D(s)$. According to [10], functions with identical images coincide for all $t > 0$ except a set of measure zero. Therefore, $w(t)$ and $d(t)$ coincide for all $t > 0$ except a set of measure zero. The asymptotic convergence of the function $e_y(t)$ will be written as $w(t) \to d(t)$.

Clearly, the converse is true: if the system is not strongly observable, its transfer function will have a zero, $H(s) = 0$ for some s. Then equality (3.4) will hold for $d(t) + e^{at}$ with $s = a$. In this case, the fault will be reconstructed within the exponent.

4. SOLUTION OF THE AUXILIARY PROBLEM

Here is its solution. For problem (3.1) and (3.2), the Hamiltonian has the form

$$H = \frac{1}{2}(z - x_*)^{\mathrm{T}} C_*^{\mathrm{T}} Q C_*(z - x_*) + \frac{1}{2} w^{\mathrm{T}} R w + \lambda^{\mathrm{T}}(\overline{A}_* z + \overline{J}_* y + D_* w + B_* u),$$

where $\overline{A}_* = A_* - KC_*$ and $\overline{J}_* = J_* + KR_*$. The optimal control law is given by

$$\frac{\partial H}{\partial w} = 0 \Rightarrow Rw + D_*^{\mathrm{T}}\lambda = 0 \Rightarrow w = -R^{-1} D_*^{\mathrm{T}} \lambda. \tag{4.1}$$

The state and conjugate variables satisfy the equations

$$\begin{aligned} \dot{z}(t) &= \frac{\partial H}{\partial \lambda} = \overline{A}_* z + \overline{J}_* y + D_* w + B_* u = \overline{A}_* z + \overline{J}_* y - D_* R^{-1} D_*^{\mathrm{T}} \lambda + B_* u, \\ z(t_0) &= \Phi x_0, \\ \dot{\lambda}(t) &= \frac{\partial H}{\partial z} = -\overline{A}_*^{\mathrm{T}} \lambda - C_*^{\mathrm{T}} Q C_* z + C_*^{\mathrm{T}} Q y_*. \end{aligned}$$

We write the latter relations in matrix form:

$$\begin{gathered} \begin{pmatrix} \dot{z}(t) \\ \dot{\lambda}(t) \end{pmatrix} = \begin{pmatrix} \overline{A}_* & -D_* R^{-1} D_*^{\mathrm{T}} \\ -C_*^{\mathrm{T}} Q C_* & -\overline{A}_*^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} z(t) \\ \lambda(t) \end{pmatrix} + \begin{pmatrix} B_* \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} \overline{J}_* y(t) \\ C_*^{\mathrm{T}} Q y_*(t) \end{pmatrix}, \\ z(t_0) = \Phi x_0. \end{gathered} \tag{4.2}$$

Equation (4.2) can be considered a diagnostic observer. By integrating (4.2) in forward time, it is possible to find and then reconstruct based on (4.1) the function describing the fault:

$$w(t) = -R^{-1} D_*^{\mathrm{T}} \lambda(t) \to d(t). \tag{4.3}$$

An open issue is the choice of initial conditions for the conjugate variable when integrating (4.2). Since the initial conditions are unknown, we introduce the following Riccati transformation [10] to find the control law:

$$z(t) = M(t)\lambda(t) + g(t), \tag{4.4}$$

where $M(t)$ and $g(t)$ are a nonsingular matrix and some vector function, respectively. Differentiating (4.4) and performing several transformations yield

$$\begin{aligned}&\left(-\dot{M}(t) + \overline{A}_* M(t) + M(t)\overline{A}_*^{\mathrm{T}} - D_* R^{-1} D_*^{\mathrm{T}} + M(t) C_*^{\mathrm{T}} Q C_* M(t)\right)\lambda(t)\\ &= \dot{g}(t) - \overline{A}_* g(t) - \overline{J}_* y(t) - B_* u(t) - M(t) C_*^{\mathrm{T}} Q C_* g(t) + M(t) C_*^{\mathrm{T}} Q y_*(t).\end{aligned}$$

This relation must hold for any $\lambda(t)$; hence, we obtain the equations

$$\begin{aligned}\dot{M}(t) &= \overline{A}_* M(t) + M(t)\overline{A}_*^{\mathrm{T}} - D_* R^{-1} D_*^{\mathrm{T}} + M(t) C_*^{\mathrm{T}} Q C_* M(t),\\ \dot{g}(t) &= \overline{A}_* g(t) + \overline{J}_* y(t) + B_* u(t) + M(t) C_*^{\mathrm{T}} Q C_* g(t) - M(t) C_*^{\mathrm{T}} Q y_*(t).\end{aligned} \tag{4.5}$$

For $t = t_0$, it follows from (4.4) that $z(t_0) = M(t_0)\lambda(t_0) + g(t_0)$. Since $\lambda(t_0)$ is unknown, the initial conditions will be satisfied by letting $M(t_0) = 0$ and $z(t_0) = g(t_0)$. Substituting (4.4) into (4.3) finally gives

$$w(t) = -R^{-1} D_*^{\mathrm{T}} M^{-1}(t)(z(t) - g(t)). \tag{4.6}$$

The ultimate expression for the desired observer has the form

$$\begin{aligned}\dot{z}(t) &= \overline{A}_* z(t) - D_* R^{-1} D_*^{\mathrm{T}} M^{-1}(t)(z(t) - g(t)) + \overline{J}_* y(t) + B_* u(t),\\ z(t_0) &= \Phi x(t_0),\\ y_z(t) &= C_* z(t).\end{aligned} \tag{4.7}$$

Here, $M(t)$ and $g(t)$ are determined from equations (4.5) with the initial conditions $M(t_0) = 0$ and $z(t_0) = g(t_0)$. On an infinite time interval, when system (3.1) is controllable and observable, the solution of equation (4.5) will tend to the steady-state value $\overline{M}$ as $t \to \infty$, which is the solution of the algebraic equation [11–13]

$$\overline{A}_* \overline{M} + \overline{M}\overline{A}_*^{\mathrm{T}} - D_* R^{-1} D_*^{\mathrm{T}} + \overline{M} C_*^{\mathrm{T}} Q C_* \overline{M} = 0;$$

the function $g(t)$ from the second equation in (4.5) will tend to the bounded solution $\overline{g}(t)$ of the differential equation

$$\dot{\overline{g}}(t) = (\overline{A}_* + \overline{M} C_*^{\mathrm{T}} Q C_*)\overline{g}(t) + \overline{J}_* y(t) + B_* u(t) - \overline{M} C_*^{\mathrm{T}} Q y_*(t) \tag{4.8}$$

with the initial conditions $\overline{g}(t_0) = z(t_0)$. The desired observer on an infinite time interval takes the form (4.7), where $M(t)$ and $g(t)$ are replaced by $\overline{M}$ and $\overline{g}(t)$.

The convergence of $g(t)$ to the bounded solution $\overline{g}(t)$ is immediate from the following considerations. Multiplying the equation for M (4.5) by -1 on the left and right and denoting $P = -M$, we obtain the Riccati equation, which typically arises in optimal estimation problems [14]. Under the conditions $R > 0$, $Q > 0$, and the observability of system (3.1), the solution of this equation is known to converge to the steady-state solution $\overline{P}$ representing the unique positive definite solution of the algebraic Riccati equation $\overline{A}_* \overline{P} + \overline{P}\overline{A}_*^{\mathrm{T}} + D_* R^{-1} D_*^{\mathrm{T}} - \overline{P} C_*^{\mathrm{T}} Q C_* \overline{P} = 0$, and $\overline{A}_* - \overline{P} C_*^{\mathrm{T}} Q C_*$ is a Hurwitz matrix. Thus, due to $\overline{P} > 0$, $P \to \overline{P}$, and $P = -M$, we obtain $M \to \overline{M}$, $\overline{M} < 0$, and the matrix $\overline{A}_* + \overline{M} C_*^{\mathrm{T}} Q C_*$ will be Hurwitz as well. With the error $e_g(t) = g(t) - \overline{g}(t)$, from (4.5) and (4.8) it follows that $\dot{e}_g(t) = (\overline{A}_* + \overline{M} C_*^{\mathrm{T}} Q C_*) e_g(t)$. Since $\overline{A}_* + \overline{M} C_*^{\mathrm{T}} Q C_*$ is a Hurwitz matrix, we have $e_g(t) \to 0$ and $g(t) \to \overline{g}(t)$ as $t \to \infty$.

5. SIMULATION RESULTS

Let us construct an observer for a two-wheeled inverted pendulum (TWIP) robot with self-balancing [15]. The kinematic diagram of this robot is shown in Fig. 1. The mathematical model of the TWIP robot linearized at the equilibrium takes the form (1.1) with the following notations:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_2 & b_2 \\ 0 & 0 \\ b_4 & b_4 \\ 0 & 0 \\ b_6 & -b_6 \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ b_2 \\ 0 \\ b_4 \\ 0 \\ -b_6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

$x = (x_l\ \dot{x}_l\ \vartheta\ \dot{\vartheta}\ \psi\ \dot{\psi})$ and $u = (T_L\ T_R)$ are the state and control vectors, respectively; T_L and T_R are the left and right wheel torques, respectively; x_l is the linear displacement; ϑ and ψ are the pitch and heading angles, respectively; $y = (x_l\ \vartheta\ \psi)$ is the system output; $d(t) = \tilde{T}_L$ is an unknown additional torque applied to the left wheel to be determined; $\rho(t) = \tilde{T}_R$ is an unknown additional torque applied to the right wheel. The coefficients a_2, a_4, b_2, b_4, and b_6 can be found using the expressions below [15]:

$$a_2 = -m_B^2 g l^2/\mu_1, \quad a_4 = \left(m_B + 2m_W + \frac{2J}{r^2}\right) m_B g l/\mu_1,$$

$$b_2 = \left((I_2 + m_B l^2)/r + m_B l\right)/\mu_1, \quad b_4 = -\left(\frac{m_B l}{r} + m_B + 2m_W + \frac{2J}{r^2}\right)/\mu_2,$$

$$b_6 = -\frac{d}{r\mu_2},$$

$$\mu_1 = \left(m_B + 2m_W + \frac{2J}{r^2}\right)(I_2 + m_B l^2) - m_B^2 l_2,$$

$$\mu_2 = I_3 + 2K_* + 2\left(m_W + \frac{J}{r^2}\right) d^2.$$

The parameter values of the robot are combined in the table.

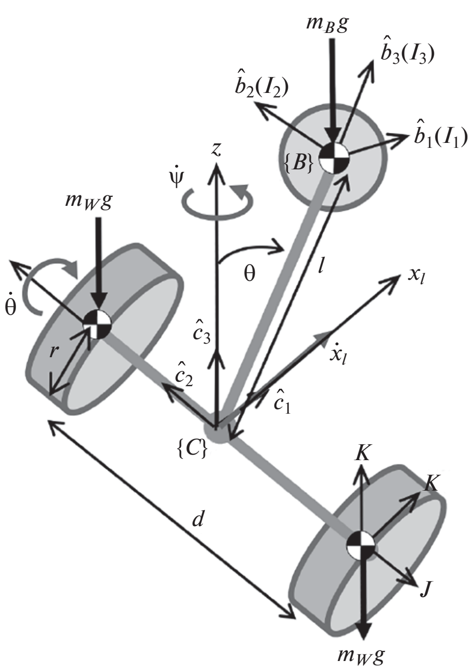

Fig. 1. The kinematic configuration of a TWIP robot.

Table

Notation	Description	Value
m_B	Weight of the pendulum body	45 kg
m_W	Wheel weight	2 kg
l	Pendulum length	0.135 m
r	Wheel radius	8 in
d	Wheel-to-wheel spacing	0.6 m
I_1, I_2, I_3	The moments of inertia of the pendulum body relative to the axes X, Y, Z	1.9; 2.1; 1.6 kg*m^2
K_*, J	The moments of inertia of the pendulum wheels relative to the vertical axis and the wheel rotation axis	0.04; 0.02 kg*m^2

To find the matrices of the reduced model (2.1), it is necessary to solve equation (2.5). For $k = 1$, it takes the form

$$(-j_1 \;\; r_1 \;\; -j_2 \;\; r_2 \;\; -j_3 \;\; r_3 \;\; 0) = 0,$$

where j_i are elements of the vector $J_* = (j_1 \;\; j_2 \;\; j_3)$ and r_i are elements of the vector $r = (r_1 \; r_2 \; r_3)$. Obviously, this equation possesses the trivial solution only. For $k = 2$, equation (2.5) takes the form

$$(-j_{21} \;\; -j_{11} \;\; a_2r_1 - j_{22} + a_4r_2 \;\; -j_{12} \;\; -j_{23} \;\; -j_{31} \;\; 0 \;\; b_2r_1 + b_4r_2 - b_6r_3) = 0,$$

where j_{ik} are elements of the matrix J_* of dimensions 2×3. In this case, the vector R_* can be chosen as $R = (0 \;\; b_6 \;\; b_4)$; then the matrix J_* becomes

$$J_* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_4b_6 & 0 \end{pmatrix}.$$

The next step is to find the matrices Φ and B_* using the expressions (2.4) and (2.3):

$$\Phi = \begin{pmatrix} 0 & 0 & b_6 & 0 & b_4 & 0 \\ 0 & 0 & 0 & b_6 & 0 & b_4 \end{pmatrix}, \quad B_* = \Phi B = \begin{pmatrix} 0 & 0 \\ 2a_4b_6 & 0 \end{pmatrix}.$$

Thus, the reduced robot model (2.6), sensitive to the function $d(t)$ and insensitive to the function $\rho(t)$, has the form

$$\begin{aligned} \dot{x}_{*1}(t) &= x_{*2}(t) + k_1r_*(t), \\ \dot{x}_{*2}(t) &= a_4b_6y_2(t) + 2a_4b_6T_L(t) + k_2r_*(t), \\ r_*(t) &= b_6y_2(t) + b_4y_3(t) - x_{*1}(t). \end{aligned}$$

The diagnostic observer for fault identification is given by (4.6)–(4.8), where $R = 10^{-2}$, $Q = 10^{20}$, and $K = (1 \;\; 1)^T$. Figure 2 shows the structural diagram of this observer.

Assume that the fault (an additional moment applied to the left wheel) is a rectangular pulse with a duration of 4 s that appears at $t = 2$.

Having adjusted the observer parameters, we obtain the identification result in Fig. 3. Clearly, the observer design approach provides an acceptable result. In addition, the graphs of the model state and observer states and observation errors can be found in Figs. 4 and 5, respectively.

Note that the quality of identification based on the optimal observer (4.6)–(4.8) depends on the choice of the penalty matrices Q and R and the matrix K. When selecting them, it is recommended to use the following considerations. The cross relations between the output and fault variables are reflected in the off-diagonal elements of these matrices. In the absence of information on such

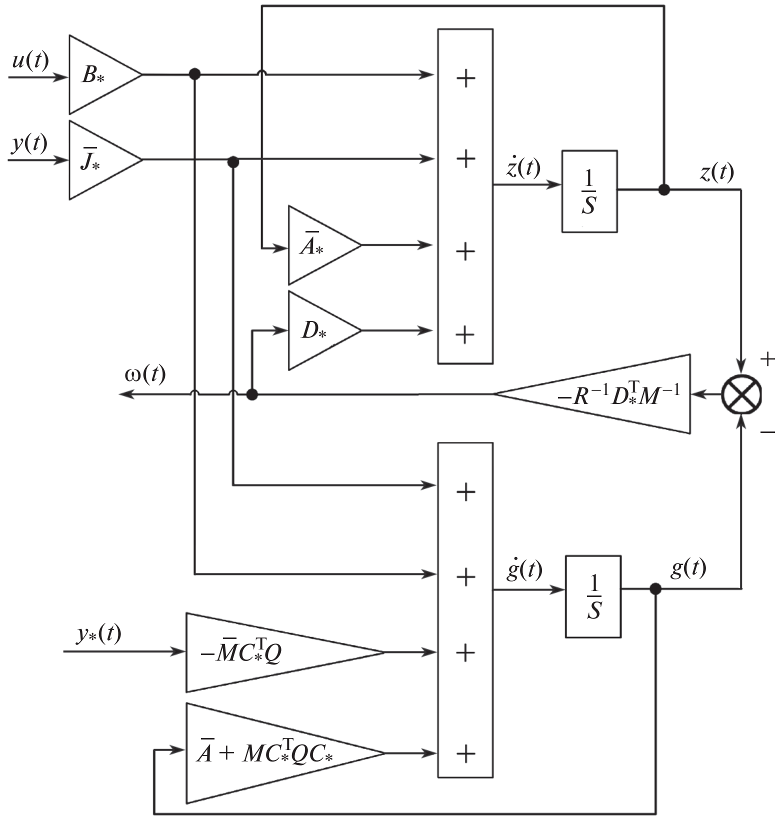

Fig. 2. Structural diagram.

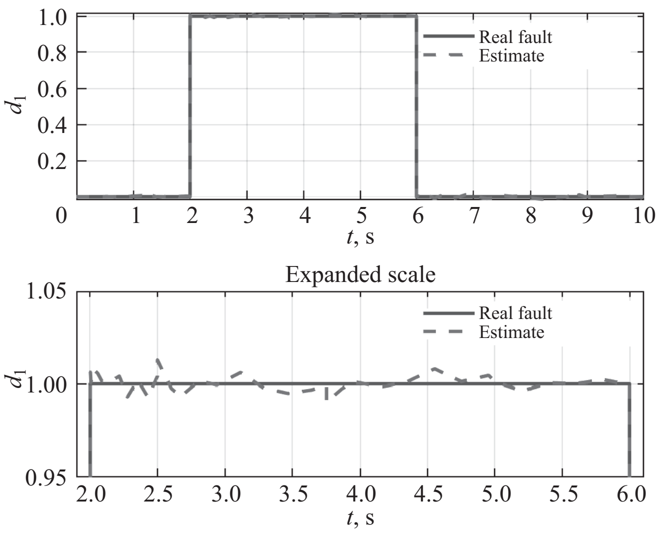

Fig. 3. Operation of the observer under a rectangular pulse fault.

relations, the diagonal form of the matrix R is recommended. The same recommendation applies to the matrix Q in the case of no cross relations between the observed outputs. If the resulting fault estimate has a large value, it is required to reduce the corresponding diagonal elements of the matrix R. Given large values of the residual $e(t) = x_*(t) - z(t)$, the elements of the matrix Q must be increased. The coefficients of the matrix K are assigned to ensure a higher performance of the observer.

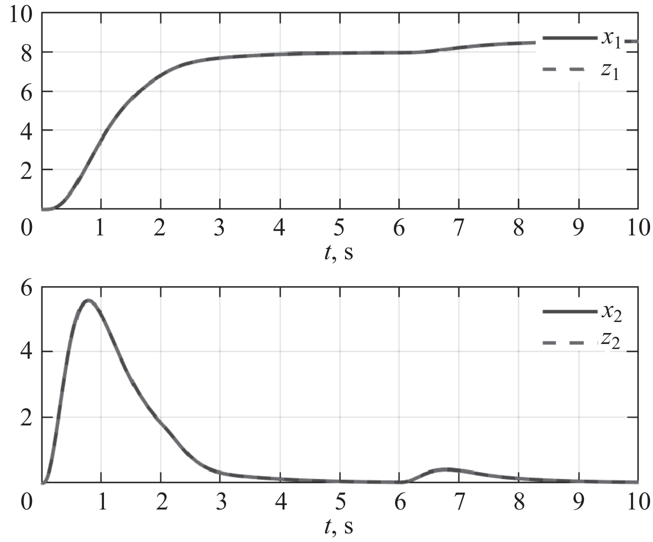

Fig. 4. The graphs of system state and diagnostic observer.

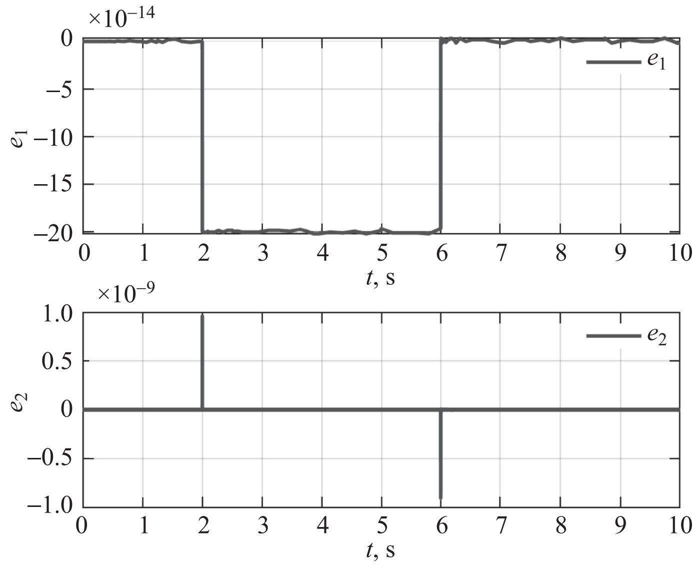

Fig. 5. The graphs of observation errors $e(t) = x_*(t) - z(t)$.

6. CONCLUSIONS

In this paper, we have estimated (identified) faults in systems described by linear models with constant coefficients under exogenous disturbances. In contrast to well-known methods based on sliding mode observers, the approach developed above expands the class of systems for which identification can be performed: the method of constructing sliding mode observers imposes restrictions on the systems for fault identification.

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